

Differentiation Formulas

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|--|---|
| 1. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$ | 2. $(kf(x))' = kf'(x)$ |
| 3. $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$ | 4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ |
| 5. $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ | 6. $\frac{d}{dx}(x^n) = nx^{n-1}$ |
| 7. $\frac{d}{dx}(e^x) = e^x$ | 8. $\frac{d}{dx}(a^x) = a^x \ln a \quad (a > 0)$ |
| | 9. $\frac{d}{dx}(\ln x) = \frac{1}{x}$ |
| 10. $\frac{d}{dx}(\sin x) = \cos x$ | 11. $\frac{d}{dx}(\cos x) = -\sin x$ |
| | 12. $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$ |

A Short Table of Indefinite Integrals

Basic Functions

- | | |
|---|--|
| 1. $\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$ | 5. $\int \sin x dx = -\cos x + C$ |
| 2. $\int \frac{1}{x} dx = \ln x + C$ | 6. $\int \cos x dx = \sin x + C$ |
| 3. $\int a^x dx = \frac{1}{\ln a} a^x + C, \quad a > 0$ | 7. $\int \tan x dx = -\ln \cos x + C$ |
| 4. $\int \ln x dx = x \ln x - x + C$ | |

I. Products of e^x , $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) dx = \frac{1}{a^2 + b^2} e^{ax} [a \sin(bx) - b \cos(bx)] + C$
9. $\int e^{ax} \cos(bx) dx = \frac{1}{a^2 + b^2} e^{ax} [a \cos(bx) + b \sin(bx)] + C$
10. $\int \sin(ax) \sin(bx) dx = \frac{1}{b^2 - a^2} [a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \quad a \neq b$
11. $\int \cos(ax) \cos(bx) dx = \frac{1}{b^2 - a^2} [b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \quad a \neq b$
12. $\int \sin(ax) \cos(bx) dx = \frac{1}{b^2 - a^2} [b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

13. $\int x^n \ln x dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1$
14. $\int p(x)e^{ax} dx = \frac{1}{a} p(x)e^{ax} - \frac{1}{a} \int p'(x)e^{ax} dx$
 $= \frac{1}{a} p(x)e^{ax} - \frac{1}{a^2} p'(x)e^{ax} + \frac{1}{a^3} p''(x)e^{ax} - \dots$
 (+ - + - ...)

(signs alternate)

$$15. \int p(x) \sin ax \, dx = -\frac{1}{a} p(x) \cos ax + \frac{1}{a} \int p'(x) \cos ax \, dx$$

$$= -\frac{1}{a} p(x) \cos ax + \frac{1}{a^2} p'(x) \sin ax + \frac{1}{a^3} p''(x) \cos ax - \dots$$

(- + + - - + + ...)

(signs alternate in pairs after first term)

$$16. \int p(x) \cos ax \, dx = \frac{1}{a} p(x) \sin ax - \frac{1}{a} \int p'(x) \sin ax \, dx$$

$$= \frac{1}{a} p(x) \sin ax + \frac{1}{a^2} p'(x) \cos ax - \frac{1}{a^3} p''(x) \sin ax - \dots$$

(+ + - - + + - - ...)

(signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

$$17. \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \quad n \text{ positive}$$

$$18. \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \quad n \text{ positive}$$

$$19. \int \frac{1}{\sin^m x} \, dx = \frac{-1}{m-1} \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\sin^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive}$$

$$20. \int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left| \frac{(\cos x) - 1}{(\cos x) + 1} \right| + C$$

$$21. \int \frac{1}{\cos^m x} \, dx = \frac{1}{m-1} \frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\cos^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive}$$

$$22. \int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left| \frac{(\sin x) + 1}{(\sin x) - 1} \right| + C$$

23. $\int \sin^m x \cos^n x \, dx$: If m is odd, let $w = \cos x$. If n is odd, let $w = \sin x$. If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18. If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21. If both m and n are even and negative, the substitution $w = \cos x$ converts the integral into a rational function which can be integrated by the method of partial fractions.

V. Quadratic in the Denominator

$$24. \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \frac{x}{a} + C, \quad a \neq 0$$

$$25. \int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \frac{x}{a} + C, \quad a \neq 0$$

$$26. \int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{a-b} (\ln |x-a| - \ln |x-b|) + C, \quad a \neq b$$

$$27. \int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{a-b} [(ac+d) \ln |x-a| - (bc+d) \ln |x-b|] + C, \quad a \neq b$$

VI. Integrands Involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$

$$28. \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a} + C$$

$$29. \int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$$

$$30. \int \sqrt{a^2 \pm x^2} \, dx = \frac{1}{2} \left(x \sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}} \, dx \right) + C$$

$$31. \int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left(x \sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C$$

SERIES CONVERGENCE/DIVERGENCE FLOW CHART

TEST FOR DIVERGENCE

Does $\lim_{n \rightarrow \infty} a_n = 0$?

NO

$\sum a_n$ Diverges

YES

p-SERIES

Does $a_n = 1/n^p, n \geq 1$?

YES

Is $p > 1$?

YES

$\sum a_n$ Converges

NO

$\sum a_n$ Diverges

NO

GEOMETRIC SERIES

Does $a_n = ar^{n-1}, n \geq 1$?

YES

Is $|r| < 1$?

YES

$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$

NO

$\sum a_n$ Diverges

NO

ALTERNATING SERIES

Does $a_n = (-1)^n b_n$ or $a_n = (-1)^{n-1} b_n, b_n \geq 0$?

YES

Is $b_{n+1} \leq b_n$ & $\lim_{n \rightarrow \infty} b_n = 0$?

YES

$\sum a_n$ Converges

NO

TELESCOPING SERIES

Do subsequent terms cancel out previous terms in the sum? May have to use partial fractions, properties of logarithms, etc. to put into appropriate form.

YES

Does $\lim_{n \rightarrow \infty} s_n = s$ finite?

YES

$\sum a_n = s$

NO

$\sum a_n$ Diverges

NO

TAYLOR SERIES

Does $a_n = \frac{f^{(n)}(a)}{n!} (x-a)^n$?

YES

Is x in interval of convergence?

YES

$\sum_{n=0}^{\infty} a_n = f(x)$

NO

$\sum a_n$ Diverges

NO

Try one or more of the following tests:

COMPARISON TEST

Pick $\{b_n\}$. Does $\sum b_n$ converge?

YES

Is $0 \leq a_n \leq b_n$?

YES

$\sum a_n$ Converges

NO

Is $0 \leq b_n \leq a_n$?

YES

$\sum a_n$ Diverges

NO

LIMIT COMPARISON TEST

Pick $\{b_n\}$. Does $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ finite & $a_n, b_n > 0$?

YES

Does $\sum_{n=1}^{\infty} b_n$ converge?

YES

$\sum a_n$ Converges

NO

$\sum a_n$ Diverges

INTEGRAL TEST

Does $a_n = f(n), f(x)$ is continuous, positive & decreasing on $[a, \infty)$?

YES

Does $\int_a^{\infty} f(x) dx$ converge?

YES

$\sum_{n=a}^{\infty} a_n$ Converges

NO

$\sum a_n$ Diverges

RATIO TEST

Is $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| \neq 1$?

YES

Is $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$?

YES

$\sum a_n$ Abs. Conv.

NO

$\sum a_n$ Diverges

ROOT TEST

Is $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \neq 1$?

YES

Is $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$?

YES

$\sum a_n$ Abs. Conv.

NO

$\sum a_n$ Diverges

Some Important Maclaurin Series

Remark: Make sure that you can derive ANY of the examples below by taking derivatives of $f(x)$

FUNCTION	TAYLOR SERIES ABOUT $x = 0$	INTERVAL OF CONVERGENCE
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$ $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < +\infty$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$ $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < +\infty$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < +\infty$
$f(x)$	$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$ $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$	Use Ratio Test to find the interval of convergence and always test the endpoints.
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$	$-1 < x \leq 1$
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$-1 \leq x \leq 1$
$(1+x)^p$	$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \binom{p}{n} x^n$ or $\sum_{n=0}^{\infty} \binom{p}{n} x^n$	$-1 < x < 1$
SOME COMMON SPECIAL CASES OF THE BINOMIAL SERIES		
$\frac{1}{1+x}; p = -1$	$1 - x + x^2 - x^3 + x^4 - \dots = \sum_{n=0}^{\infty} (-1)^n x^n$	$-1 < x < 1$
$\frac{1}{1-x}; p = -1$	$1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\frac{1}{(1+x)^2}; p = -2$	$1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$	$-1 < x < 1$
$\frac{1}{(1-x)^2} = \frac{1}{(x-1)^2}$	$1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} (n+1) x^n$	$-1 < x < 1$

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{1}{2} \cdot \frac{x^2}{2!} + \frac{1 \cdot 3}{2^3} \frac{x^3}{3!} - \frac{1 \cdot 3 \cdot 5}{2^4} \frac{x^4}{4!} + \dots$$

$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \dots (2n-3)}{2^n n!} x^n$$

$-1 < x < 1$

general Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$