Definition of Limits (Informal)
Let \( f(x) \) be defined on an open interval about \( x_0 \), except at \( x_0 \). If \( f(x) \) gets arbitrarily close to \( L \) for all \( x \) sufficiently close to \( x_0 \), we say that \( f \) approaches the limit \( L \) as \( x \) approaches \( x_0 \), that is
\[
\lim_{x \to x_0} f(x) = L.
\]

Definition of Limits (Formal)
Let \( f(x) \) be defined on an open interval about \( x_0 \), except at \( x_0 \). We claim
\[
\lim_{x \to x_0} f(x) = L,
\]
if, for every number \( \varepsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that for all \( x \),
\[
0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.
\]

Definition of One-sided Limits (Formal)
Right-hand limit
Let \( f(x) \) be defined on an open interval about \( x_0 \), except at \( x_0 \). We claim
\[
\lim_{x \to x_0^+} f(x) = L,
\]
if, for every number \( \varepsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that for all \( x \),
\[
x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon.
\]
Left-hand limit
Let \( f(x) \) be defined on an open interval about \( x_0 \), except at \( x_0 \). We claim
\[
\lim_{x \to x_0^-} f(x) = L,
\]
if, for every number \( \varepsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that for all \( x \),
\[
x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon.
\]

Theorem of Limits
A function \( f(x) \) has a limit as \( x \) approaches \( x_0 \) if and only if it has left-hand and right-hand limits there and these one-sided limits are equal;
\[
\lim_{x \to x_0} f(x) = L \iff \lim_{x \to x_0^-} f(x) = L \land \lim_{x \to x_0^+} f(x) = L.
\]
Definition of limit as $x$ approaches infinities

**Positive infinity**
\[
\lim_{x \to \infty} f(x) = L, \text{ if,}
\]
for every number $\varepsilon > 0$, there exists a corresponding number $M$ such that for all $x$,
\[
x > M \Rightarrow |f(x) - L| < \varepsilon.
\]

**Negative infinity**
\[
\lim_{x \to -\infty} f(x) = L, \text{ if,}
\]
for every number $\varepsilon > 0$, there exists a corresponding number $N$ such that for all $x$,
\[
x < N \Rightarrow |f(x) - L| < \varepsilon.
\]

**Definition of infinite limits**

**Positive Infinity**
\[
\lim_{x \to x_0} f(x) = \infty, \text{ if,}
\]
for every positive real number $B$, there exists a corresponding $\delta > 0$ such that for all $x$,
\[
0 < |x - x_0| < \delta \Rightarrow f(x) > B.
\]

**Negative Infinity**
\[
\lim_{x \to x_0} f(x) = -\infty, \text{ if,}
\]
for every negative real number $-B$, there exists a corresponding $\delta > 0$ such that for all $x$,
\[
0 < |x - x_0| < \delta \Rightarrow f(x) < -B.
\]

**Definition of horizontal asymptotes**
A line $y = b$ is defined as a horizontal asymptote of a function $y = f(x)$ if
\[
\lim_{x \to \infty} f(x) = b \lor \lim_{x \to -\infty} f(x) = b
\]

**Definition of vertical asymptotes**
A line $x = a$ is defined as a vertical asymptote of a function $y = f(x)$ if
\[
\lim_{x \to a^+} f(x) = \pm\infty \lor \lim_{x \to a^-} f(x) = \pm\infty
\]
Test of Continuity
A function $f(x)$ is continuous at $x = x_0$ if and only if it meets all the following conditions.
1. $f(x_0)$ exists
2. $\lim_{x \to x_0} f(x)$ exists
3. $\lim_{x \to x_0} f(x) = f(x_0)$

Note that condition 2 can be checked using the Theorem of Limits mentioned before.

**Theorem of Continuity** Students often use this theorem without realizing it.
If $g$ is continuous at the point $b$ and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g\left(\lim_{x \to c} f(x)\right)$$

example: $g(x) = x^2$, $f(x) = 3x$, at $c = 3$

$$\lim_{x \to 3} (3x)^2 = (3 \times 3)^2 = \left[\lim_{x \to 3} (3x)\right]^2$$